Proof by contraposition:

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$$

The idea is: show that $\neg Q \Rightarrow \neg P$ is true by direct proof.

Example: If m^2 is odd, then m is odd.
Remark: An integer k is odd if it is of the form $k = 2j + 1$ for some integer j.
k is even if k is of the form $k = 2j$ for some integer j.
k is even if and only if 2 divides k, and k is odd if and only if 2 does not divide k.

P:"m^2 is odd"
Q:"m is odd"
Show that $P \Rightarrow Q$ is true.
Instead we will show that the contrapositive is true: $\neg Q \Rightarrow \neg P$
"If m is even, then m^2 is even."
Prove this directly:

Suppose m is even.
Then $\exists j$ such that $m = 2j$
Then $m^2 = m * m$ by the definition of squared.
$m^2 = (2j)(2j)$ by substitution.
$m^2 = 2(2j)$ by factoring 2 out.
So m^2 is even because 2 divides m. So if m^2 is odd, then m is odd by contraposition.

Proof by Contradiction:
Recall $(P \Rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$ is a tautology.
We want to show $P \Rightarrow Q$, which is the same as showing $\neg(P \wedge \neg Q)$ is true.
So we need to show that $P \wedge \neg Q$ is false.

How will we do this?
If $T \Rightarrow S$ is true, but you know S is false, T must be false as well.

The idea is to make $P \wedge \neg Q$ the hypothesis in an implication: $(P \wedge \neg Q) \Rightarrow S$ such that the implication is true and S is false.

Question: How do we find an S?
Answer: If R is a proposition, then $R \wedge \neg R$ is always false.
So proof by contradiction looks like this:

$$(P \wedge \neg Q) \Rightarrow (R \wedge \neg R)$$

Example: If $x = \sqrt{2}$, then x is not rational.

Facts:
1. Fundamental Theorem of Arithmetic: Every whole number is a unique product of prime numbers.
2. If m and n are whole numbers, then we can cancel common factors from $\frac{m}{n}$ and we may assume $\frac{m}{n}$ has no common factors.
3. If m^2 is a number, each of its prime factors occurs an even number of times.

Proof by contradiction:
Assume $x = \sqrt{2}$ and x is rational.
Then $\sqrt{2} = \frac{m}{n}$ by definition of rational.
Then m and n have no common factors because of fact 2.
Then $\sqrt{2}n = m$ by multiplying both sides by n.
Then $2n^2 = m^2$ by squaring both sides.
$\neg R$: But 2 occurs as a prime factor of $2n^2 = m^2$ both an even and an odd number of times, but...
R: ...the prime factorization of a number is unique.
So it is not the case that $x = \sqrt{2}$ and $\sqrt{2}$ is rational.
So $\sqrt{2}$ is irrational.