Notes for March 31st

We have lots of statements that depend on something = open statement (These are not tautologies)
Example:
$P(x) = 7x + 4 = 3$
Plug in a variable $\rightarrow$ get T or F
$P(7) \rightarrow false$

Definition: Suppose an open sentence P(x) depends on the variable x. The universally qualified statement is the statement "for all x P(x)."
Notation: ∀x = "for all x"
∀ is called the universal quantifier

Defintion: The existentially quantified statement is the statement "there exists x such that"
Notation: ∃x = "There exists x such that"
∃ is called the existential quantifier

Example:
$P(x) : 4x + 3 = 7$ (Assume universe for all real numbers)
$\forall x P(x)$ means "for all x (or for every x) $4x + 3 = 7$"
$\exists x P(x)$ means "there exists x such that $4x + 3 = 7$"

Remark: (i) Quantified statements will be either true or false -
i.e., they are propositions.
(ii) Truth value will depend on the universe -
e.g., $\forall x P(x)$ is false if the universe is all real numbers, but true if the universe is the set 1.

(Note: In general, m,n,k,are Integers, x,y are real numbers, z is complex, and f,g are functions.)

Quantifier Tautologies
1. $\sim (\forall x P(x)) \Leftrightarrow \exists x(\sim P(x))$
2. $\sim (\exists x P(x)) \Leftrightarrow \forall x(\sim P(x))$
These tautologies help us switch to and from universal and existential.

Example:
$\lim_{x\to 2} f(x) = L \Leftrightarrow$ "for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$"
Let $P(\delta) : |x - a| < \delta Q(\epsilon) : |f(x) - L| < \epsilon$
$P(\delta) \Rightarrow Q(\epsilon)$
$\forall \epsilon > 0(\exists \delta > 0(P(\delta) \Rightarrow Q(\epsilon)))$
When does a limit not exist?
When: $\sim (\lim_{x\to a} f(x) = L) \Leftrightarrow \sim (\forall \epsilon > 0(\exists \delta > 0(P(\delta) \Rightarrow Q(\epsilon))))$
$\Leftrightarrow (\exists \epsilon > 0(\sim (\exists \delta > 0(P(\delta) \Rightarrow Q(\epsilon)))$
$\Leftrightarrow \exists \epsilon > 0(\forall \delta > 0(\sim (P(\delta) \Rightarrow Q(\epsilon))))$

[Recall: We will use: $(P \Rightarrow Q) \Leftrightarrow \sim (P \wedge ((\sim Q))$
To form: $\sim (P \Rightarrow Q) \Leftrightarrow \sim (\sim (P \wedge (\sim Q)))$]

$\Leftrightarrow \exists \epsilon > 0(\forall \delta > 0(P(\delta) \wedge (\sim Q(\epsilon))))$
So $\sim (\lim_{x\to a} f(x) = L)$ if and only if there exists $\epsilon > 0$ such that for every $\delta > 0 |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

Basic Proof Strategies
Idea: Almost all mathematical propositions/theorems are of the form $P \Rightarrow Q$. How to show a statement of the form $P \Rightarrow Q$ is true.

1.Direct Proof:

Recall the truth table for $P \Rightarrow Q$ We will work with the cases where P is true, because Q may be either true or false in these cases.

Assume P is true. Then show (using a "proof") that Q is true.

Example: (In number theory)
We say: $a|b \Leftrightarrow \exists k such that b = ka$, where a, b, and k are integers.

(i) If $a|b$, then $a/(-b)$
Direct Proof: Assume a/b (Hypothesis)
Then there exists k such that $b = ka$ (Definition)
Then $-b = -(ka)$ (Rule of algebra = Previous Result)
So $-b = -(ka) = (-k)a$ (Associative properties = Previous Result)
Because $-b = -1(ka) = ((-1)k)a = (-k)a$
Thus, there exists a number -k such that $-b = (-k)a$ (Previous step in proof)
So $a/(-b)$ (Definition)