

Multiple Statements and Tautologies

Truth tables do not have to involve just two statements.

$$P \rightarrow (P \vee Q \vee R)$$

This may seem correct, but it doesn't mean anything based on the rules of logic.

The correct way to say this would be: $P \rightarrow (P \vee (Q \vee R))$

This is similar to $1 + 1 + 1$ seeming alright, but in reality it should be $1 + (1 + 1)$ or, by associativity $(1 + 1) + 1$

Excercise: Show that $(P \vee (Q \vee R)) \iff ((P \vee Q) \vee R)$

Now let's see if the statement said before is true;

P	Q	R	$Q \vee R$	$P \vee (Q \vee R)$	$P \rightarrow (P \vee (Q \vee R))$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	F	T

So the statement is true.

Definition: A *tautology* is a proposition which is always true. A *contradiction* is a statement which is always false. A statement is indeterminate if it is sometimes true and sometimes false.

Example: Is $P \rightarrow Q$ a tautology, a contradiction, or is it indeterminate?

P	$P \rightarrow Q$	Q
T	T	T
T	F	F
F	T	T
F	T	F

Since the statement is sometimes true and sometimes false, it is indeterminate.

Important tautologies

Now we are going to look at some important Logical Equivalences: tautologies of the form "this if and only if that."

1. $P \vee (\sim P)$

P	\vee	$(\sim P)$	P
T	T	F	T
F	T	T	F

So, either P is true or not P is true.

2. $(P \vee (Q \vee R)) \iff ((P \vee Q) \vee R)$

Do the proof yourself for practice (DIY).

3. $(P \wedge (Q \wedge R)) \iff ((P \wedge Q) \wedge R)$

(The proof of this was done as a previous exercise).

$$4. \sim(\sim P) \iff P$$

Proof:

\sim	$(\sim$	$P)$	\iff	P
T	F	T	T	T
F	T	F	T	F

So far the tautologies that we've seen are kind of trivial; basic ideas that, while important, we take for granted. Here's some more interesting stuff:

$$5. (P \rightarrow Q) \iff (\sim(P \wedge (\sim Q)))$$

Proof:

$(P$	\rightarrow	$Q)$	\iff	$(\sim$	$(P$	\wedge	$(\sim$	$Q))$
T	T	T	T	T	T	F	F	T
T	F	F	T	F	T	T	T	F
F	T	T	T	T	F	F	F	T
F	T	F	T	T	F	F	T	F

For tautologies 6 through 11, complete the proofs yourself.

The next two tautologies are known as DeMorgan's laws.

$$6. \sim(P \wedge Q) \iff \sim P \vee \sim Q$$

$$7. \sim(P \vee Q) \iff \sim P \wedge \sim Q$$

$$8. (P \wedge (P \rightarrow Q)) \rightarrow Q$$

(This is known as Modus Ponens)

$$9. ((P \rightarrow Q) \wedge (\sim Q)) \rightarrow (\sim P)$$

(Modus Tollens)

$$10. (\sim P \rightarrow (\sim Q \wedge Q)) \iff P$$

(This is called contradiction, and is very useful in some proofs).

$$11. (P \iff Q) \iff ((P \rightarrow Q) \wedge (Q \rightarrow P))$$

$$12. (P \rightarrow Q) \iff (\sim Q \rightarrow \sim P)$$

(This is called contrapositive, and is also useful in proofs.)

Proof:

$(P$	\rightarrow	$Q)$	\iff	$(\sim$	Q	\rightarrow	\sim	$P)$
T	T	T	T	F	T	T	F	T
T	F	F	T	T	F	F	F	T
F	T	T	T	F	T	T	T	F
F	T	F	T	T	F	T	T	F

Here's a famous **non**-tautology

$(P$	\rightarrow	$Q)$	\iff	$(Q$	\rightarrow	$P)$
T	T	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	F	T	F	F
F	T	F	T	F	T	F

Idea: $Q \rightarrow P$ is called the **converse** of $P \rightarrow Q$. If $P \rightarrow Q$ is true, $Q \rightarrow P$ might not be true (and vice versa).

Example: (Intermediate value theorem)

If $f(x)$ is continuous on $[a, b]$ and $f(a) > 0$ and $f(b) < 0$, then there exists c on $[a, b]$ where $f(c) = 0$.

P: f is continuous on $[a, b]$ and $f(a) > 0$ and $f(b) < 0$.

Q: There exists c in $[a, b]$ such that $f(c) = 0$.

I.V.T.: $P \rightarrow Q$ is true.

Converse: $Q \rightarrow P$. " If there exists c on $[a, b]$ with $f(c) = 0$, then f is continuous on $[a, b]$ and $f(a) > 0$ and $f(b) < 0$." This statement is indeterminate: you cannot conclude that it is always true.

Contrapositive: $(P \rightarrow Q) \iff (\sim Q \rightarrow \sim P)$, where

$\sim Q$: There isn't a c in $[a, b]$ for which $f(c) = 0$.

$\sim P$: It is not the case that (f is continuous on $[a, b]$ and $f(a) > 0$ and $f(b) < 0$).

Notice: $\sim P$ is of the form

$\sim (R \wedge (S \wedge T))$

R: f is continuous on $[a, b]$

S: $f(a) > 0$

T: $f(b) < 0$

$\sim (R \wedge (S \wedge T)) \iff (\sim R) \vee (\sim (S \wedge T)) \iff (\sim R \vee (\sim S \vee \sim T))$

by DeMorgan's laws!

Exercise: Show that $(P \rightarrow Q) \iff (\sim P \rightarrow \sim Q)$ is not a logical equivalence.

Exercise: Show that

"If $xy = 0$, then $x = 0$ or $y = 0$ "

is logically equivalent to

"if $xy = 0$ and $x \neq 0$, then $y = 0$."

Remark: This branch of logic is sometimes called *propositional logic*, because it depends on "input" propositions P, Q, ... being either true or false. But there are lots of statements in mathematics which are sometimes true and sometimes false...

Example: $4x + 3 = 7$

The "truthiness" of this depends on what x is.

Definition: A sentence whose truth value depends on a variable (or variables) is called an open sentence and is denoted by $P(x)$, $P(x, y)$, ...

Example: $P(x)$: " $4x + 3 = 7$ "

The set of x values for which the statement is true are called the *truth set*. The larger set containing the truth set is called the *universe* (which can be all real numbers, all integers, etc.)

Two open sentences are *equivalent* if they have the same truth set (in a given universe).