An Introduction to James Gregory's *Geometriae Pars Universalis*

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For mathematics educators with an interest in the history of mathematics the importance of original sources is clear. In contrast to the polished presentations of mathematics found in textbooks, original sources show mathematics as it is in real–life: long–struggling thinkers wrestling with ill-defined concepts to arrive at imperfect results. Indeed, as Reinhard Laubenbacher and David Pengelley have written, when students use original sources, mathematics is "seen as an evolving human endeavor, its theorems the result of genius struggling with the mysteries of the mathematical universe, rather than an unmotivated, ossified edifice of axioms and theorems handed down without human intervention" [**4**, p. 257].

Nowhere is this contrast between mathematics as it was developed and mathematics as it is usually taught more clear than in calculus. In most cases, even the most diligent student will leave a calculus class with only the article of faith that "Newton and Leibniz invented calculus". Students who go on to an advanced calculus course will most likely see only the axiomatic underpinnings of calculus—i.e., calculus as it was known in the nineteenth century. In short, it's up to the history of mathematics teacher to introduce students to the intriguing and intellectually demanding story of calculus as it unfolded throughout the centuries. The goal of this article is to provide an introduction to the *Geometriae Pars Univeralis*, a work which brings together in one place many seventeenth century techniques for solving calculus problems, but which has been largely overlooked by mathematics educators for use as an original source.

James Gregory, the author of the *Geometriae Pars Universalis*, has certainly not himself been overlooked. Even students in calculus are taught "Gregory's series", the power series expansion of the arctangent. Students in history of mathematics courses will also learn that Gregory was a truly exceptional and original thinker. Among the things that are ascribed to him in history texts:

- He "described the reflecting telescope now known by his name" [**2**, p. 299].
- He was "one of the first to distinguish between convergent and divergent series" [**2**, p. 299].
- In addition to the series expansion for arctangent, he also found series expansions for tangent and secant [**2**, p. 299].
- He discovered the Taylor series 40 years before Taylor [**3**, p. 494].
- He "gave an ingenious but unsatisfactory proof that Euclidean quadrature of the circle is impossible" [**2**, p. 299].
- He presented a version of the fundamental theorem of calculus [**3**, p. 499].

Much of the credit for helping mathematicians to recognize the importance of Gregory's original mathematical contributions goes back to Herbert Turnbull, whose 1939 *James Gregory Tercentenary Memorial Volume* [**7**] revealed through a detailed study of Gregory's personal notes and correspondences that he was in possession of many sophisticated techniques for dealing with series expansions. More recently, Gregory's original work has also been the subject of a Ph.D. thesis by Antoni Malet (see [**6**]).

In contrast, among the body of Gregory's work the *Geometriae Pars Universalis* is often dismissed by historians as being the most unoriginal. As Gregory himself states in the introduction to the work, "Let the reader who has compared this work with others judge what is mine and what belongs to another" [**5**]. But this shortcoming in the eyes of mathematical historians also makes it a rare find for mathematics educators. To see why this is the case, it is important to understand the background of the work. Gregory spent most of his life in Scotland on the fringe of the academic world. Born in 1638 near

Aberdeen, he was educated at the University of Aberdeen. Eventually, he would hold a chair in mathematics, first at the University of St. Andrews from 1669 to 1674 and then at the University of Edinburgh until his death in 1675. (See [**8**] for the details of Gregory's life.) The one noteworthy exception to this life spent outside the mainstream of the intellectual world was a trip Gregory took to the continent during 1664-1667. It was during this trip, when he studied in Italy with Stefano degli Angeli (a student of Cavalieri), that Gregory became acquainted with the mathematical developments occurring on the continent. The *Geometriae Pars Universalis*, published in Padua in 1668, presents a survey of the material Gregory absorbed on this trip. Consequently, in 70 propositions spread over 132 pages, the *Geometriae Pars Universalis* provides an example of a book which summarizes the techniques of calculus prior to the discoveries of Newton and Leibniz. Moreover, the material on finding tangents, arclengths, and areas—that is, the material most accessible to students in history of mathematics courses—can be found in just the first eleven propositions.

Our goal is to look in detail at several examples from these first eleven propositions. In particular, we will derive a modern interpretation of several of his results in order to show the broad range of mathematical ideas contained in these few pages. Our approach is essentially the same as A. Prag in [**7**] or Margaret Baron in [**1**], both of whom have carried out a similar analysis of the *Geometrae Pars Universalis*.

The first major result in the work is Proposition Two [**5**]:

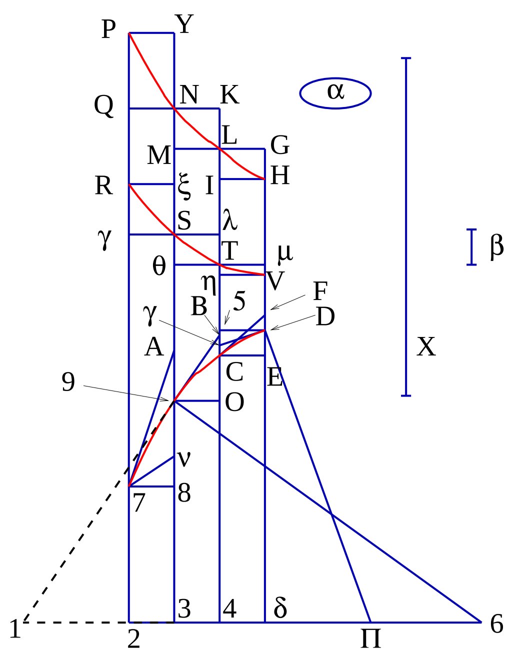
Proposition Two. Theorem

Let 79CD be any simple and non-winding curve (if indeed it were winding, it is reasonable to divide it into many simple curves) above which is imagined the surface of a right cylinder whose altitude is the line X. From some point of the curve, say, 9, let a line 93 be dropped down perpendicular to a line, say, 26. Also, let a line 96 cutting the curve normally in the point 9 be drawn to the line 26. Let the line 39 be extended to a point S such that 3S is equal to the line 96. It should likewise be supposed that this is done in all points of the curve 79CD in such a way that from the lines perpendicular to the curve extended normally to the line

26 through their own points a region RVδ2 composed of the curve RSTV and the lines R2, Vδ, and δ2 is produced. Next, let the altitude X of the cylinder be to 3N as 93 is to S3. Likewise, we suppose that this is done in all the other lines of the region RVδ2 perpendicular to the line 2δ, so that a region PHδ2 comprised of the curve PH and the lines Hδ, P2, and 2δ is filled out.

I say that the second curved area PHδ2 is equal to the surface of the right cylinder whose base is the given curve 79CD and which has line X as the altitude.

The length of the statement of this proposition will be intimidating for any student who has never wrestled with an original source, but it will not be difficult for any student with a background in calculus and geometry to understand the meaning behind it. Gregory sheds some light on the material by providing (as he does for most of his results) a figure:



(We've added the tangent line 91 to the curve 79CD at the point 9 and the subtangent 13 over the same interval, but otherwise the figure is identical to the one Gregory provides.) Note that the biggest difficulty in understanding this figure is that Gregory is trying to work in three dimensions. The segment X should be thought of as projecting out perpendicularly to the page at every point on the curve $79CD$, thus forming a curved rectangular figure with altitude X perpendicular to the curve.

A large part of the difficulty in understanding this proposition (aside from Gregory's unfortunate tendency to use numbers as variables to label points in his figure) is the fact that Gregory is hindered by not having the concept of a function. With the idea of a function in hand, it's not difficult to approach this proposition from a modern perspective. His first sentence essentially states that $79CD$ is the graph of a convex function, say, $f(x)$. The second sentence states that the line 26 should be thought of as the x–axis, with 3 being the point x on this axis. In the third sentence, the line 96 denotes the segment normal to the curve at 3 on the x–axis at 6. The next four sentences define the curve $PNLH$, which we shall denote by $g(x)$, by the relation $\frac{f(x)}{96} = \frac{X}{g(x)}$, where X is the altitude of the rectangular surface described above.

The key to understanding the conclusion ("I say that…") of the proposition in modern terms is deriving a formula depending on x for the segment 96 and arriving at an explicit formula for $g(x)$ using the relation given above. If 91 is the tangent line to $f(x)$ at x, then the segment 91 is perpendicular to 96 and consequently the triangles $\triangle 931$, $\triangle 691$, and $\triangle 639$ are all similar. Hence, since the the segment $93 = f(x)$, comparing corresponding sides yields $\frac{f(x)}{31} = \frac{36}{f(x)}$. Since $f'(x)$ is the slope of the tangent line at x, we also know that $f'(x) = \frac{f(x)}{31}$. Combining these two equations, we conclude that $36 = \frac{f(x)^2}{31} = f(x)f'(x)$. Applying the Pythagorean theorem to the triangle with sides the segments labeled 96, 39, and 36, implies

$$96 = \sqrt{39^2 + 36^2} = \sqrt{f(x)^2 + (f(x)f'(x))^2} = f(x)\sqrt{1 + f'(x)^2}$$

so that the defining relation $\frac{f(x)}{96} = \frac{X}{g(x)}$ for $g(x)$ can be rewritten as $\frac{1}{\sqrt{1 + (f'(x))^2}} = \frac{X}{g(x)}$ and hence $g(x) = X\sqrt{1 + (f'(x))^2}$. The conclusion of the proposition states that the area under the graph of $g(x)$ over the interval $[2, \delta]$ is equal to the area of the rectangular shape in $\mathbf{R}^3$ with height X and base the graph of $f(x)$ over the interval $[2, \delta]$. In other words,

$$\int_2^\delta X\sqrt{1 + (f'(x))^2}\, dx = X \cdot \text{arclength}(f)$$

or

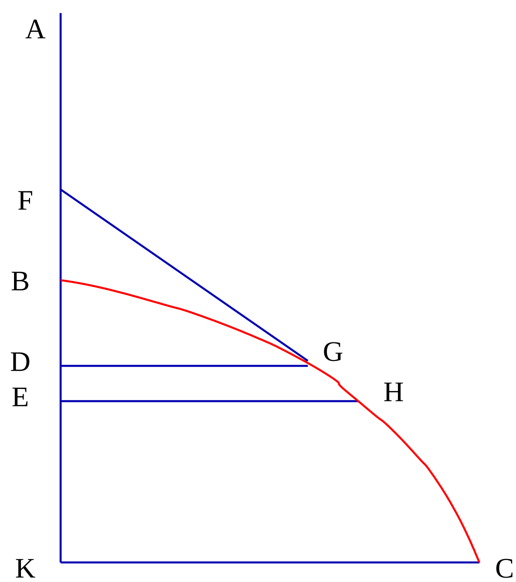
$$\int_2^\delta \sqrt{1 + (f'(x))^2}\, dx = \text{arclength}(f).$$

So already in the second proposition, Gregory has derived a result which is equivalent to the modern formula for arclength. Incidentally, the vestiges of Gregory's proof can be seen in the figure above. It is an application of the method of exhaustion and is immediately accessible to any student who is familiar with the techniques of Archimedes. For a contradiction, Gregory assumes that the two areas are different and denotes their difference by α (the plane region pictured in the figure). He then proceeds to inscribe and circumscribe a finite number of rectangles (on the subintervals represented by $[2, 3]$, $[3, 4]$, $[4, \delta]$ given in the figure) and arrive at a contradiction using a standard similarity argument.

In addition to classical Archimedean techniques for finding areas, Gregory is also interested in finding tangents. In several cases, he proves his results using geometrical techniques similar to those found in Apollonius. He also provides an example of Fermat's analytic technique in Proposition Seven [**5**]:

Proposition Seven. Theorem

To draw a line tangent to a given curve at a given point of the curve if the curve is from the category which Descartes calls Geometrical.



Unlike all of Gregory's other results, this result is proved for a specific function. As Gregory states [**5**]:

> Let the curve BHC be a hyperbola whose diameter is the line AK and whose ordinates EH and KC are of such a nature that the solid formed by the square on BE together with AE is to the solid formed by the square on BK together with AK as the cube on EH is to the cube on KC.

For a modern interpretation, a change of notation is in order: Gregory intends for AK, KC, BK, and AB to be fixed constants and the lengths BE and EH to be variable. Let $AB = a$, $BE = x$, and $EH = y$. Then the proportion described above can be rewritten as $\dfrac{x^2(a+x)}{BK^2 \cdot AK} = \dfrac{y^3}{KC^3}$ or $x^2(a+x) = ey^3$, where $e = \dfrac{BK^2 \cdot AK}{KC^3}$. Note that this is the equation of an algebraic curve having a shape similar to a hyperbola, rather than the equation of a hyperbola. Like his peers, Gregory is interested in the length of the subtangent EF, rather than finding the slope of the tangent line HF. For us, this would involve differentiating implicitly to find $\dfrac{dy}{dx} = \dfrac{2xa + 3x^2}{3ey^2}$. If z is the length of the subtangent, then $\dfrac{dy}{dx} = \dfrac{y}{z}$, so that $z = \dfrac{3x(a+x)}{2a+3x}$.

If Fermat's method of tangents is essentially the modern method of finding tangents, then Gregory's proof is not substantially different from a modern proof (minus the notion of the limit). To start things off, he finds another way to write the equation of the curve [**5**]:

> Let the given AB be a, let BE be b, and let the ratio of the solid from the square on BE together with AE be to the cube on EH as a^3 is to c^3. It is desired to find a point F so that the line FH touches the hyperbola in the point H.

Here c is defined as follows: Using the variable x for Gregory's b as above, $\dfrac{x^2(a+x)}{BK^2 \cdot AK} = \dfrac{y^3}{KC^3}$ can be rewritten as $\dfrac{x^2(a+x)}{y^3} = \dfrac{BK^2 \cdot AK}{KC^3}$. The right-hand side of this is a constant, so c is then defined as a constant such that $\dfrac{x^2(a+x)}{y^3} = \dfrac{a^3}{c^3}$. Gregory then solves for y in this equation to arrive at the function $y(x) = \dfrac{\sqrt[3]{c^3x^2(a+x)}}{a}$. The next step is to actually begin the analysis. He lets DE be o and, supposing that the tangent line has already been drawn, he denotes the subtangent FE to $y(x)$ at x by z so that $DF = z - o$. Then "since we suppose that the ordinate DG falls upon the curve in the same point G where the line FH runs into the same curve (if it is able to be done)", $DG = y(x - o)$ and by similar triangles $\dfrac{EH}{EF} = \dfrac{DG}{DF}$. Thus, $EH \cdot DF = DG \cdot EF$.

What happens next is a gruesome piece of algebra: Plug in $z - o$ for DF and z for EF as well as the formulas for $y(x)$ and $y(x-o)$ for EH and DG, respectively. Then cube both sides, cross multiply, cancel like terms, and factor out the o common to what remains on both sides. After "rejecting the quantities in which o or a power of o is found", the result is:

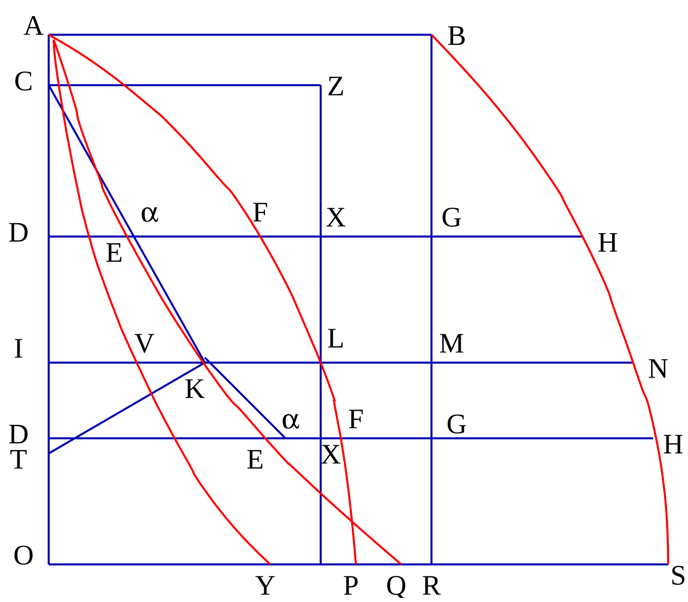
$$3ac^3x^2z^2 + 3c^3x^3z^2 = 2ac^3xz^3 + 3c^3x^2z^3$$

Canceling common factors and solving for z, the result is $z = \dfrac{3ax + 3x^2}{2a + 3x}$, as above.

Perhaps Gregory's most stunning observation is found buried in the proof of a result which looks somewhat uninteresting by today's standards [**5**]:

Proposition Six. Theorem

To find a curve which has the same ratio to its axis as any exhibited figure has to a rectangle inscribed in it and applied to a given line or axis of the sought–after curve.



Following Gregory's figure, we take the vertical axis $A0$ as the axis of the independent variable. In modern terms, if $f(y)$ is a given function (with graph $BHNS$), the goal of this proposition is to construct a function $g(y)$, whose arclength has the same ratio to its axis as the area under the graph of $f(y)$ has to an inscribed rectangle with the same base. In other words, given $f(y)$, the goal is to find a function $g(y)$ such that

$$\frac{\text{arclength}(g(y))}{y} = \frac{\int_0^y f(t)\,dt}{cy}$$

where y is arbitrary and c is the length of the segment AB. Using the definition of the arclength and canceling the common y in the denominator of both sides, we arrive at

$$\int_0^y \sqrt{1+(g'(t))^2}\,dt = \frac{1}{c}\int_0^y f(t)\,dt$$

There's nothing terribly intriguing in this statement, but that's not the case for the proof. To begin with, he constructs the function $g(y)$ in two steps. In the first step, he constructs a function $h(y)$ via the defining equation $f(y)^2 = c^2 + h(y)^2$. (In terms of the figure, $AI = y$, $IM = BA = c$, and $h(y)$ is the curve $AFLP$.) Then $g(y)$, which has graph AKQ, is defined by $cg(y) = \int_0^y h(t)\,dt$. Without going into details, the crucial step in his proof is to show that if C is defined by the equation $\frac{IK}{IC} = \frac{IL}{IM}$, then the line CK is tangent to $g(y)$. Looked at in modern terms, this observation is quite stunning. For if CK is tangent to $g(y)$, then $g'(y) =$ the slope of CK $= IK/IC$, is equal to IL/IM. Since $IL = h(y)$ and $IM = c$, this means that $g'(y) = \frac{h(y)}{c}$, by definition of the derivative. In other words, Gregory has concluded that if $cg(y) = \int_0^y h(t)\,dt$, then $cg'(y) = h(y)$. If this doesn't seem familiar, it should: It's the fundamental theorem of calculus.

As noted above, historians have tended to ignore the *Geometriae Pars Universalis* because it does not highlight Gregory's creative work as a mathematician. Even among the propositions we have selected, Gregory's indebtedness is apparent. The first arclength problems had been solved using similar methods ten years earlier by William Neile and Hendrick van Heuraet [**1**, pp. 223–4]. The method of tangents discussed in Proposition

Seven is likewise due completely to Fermat, who had first presented it in 1637 [**1**, pp. 166–170]. Even the proof of the fundamental theorem of calculus found in Proposition Six, which has been hailed as "the first published statement, in geometrical form, of what is now termed the fundamental theorem of calculus" [**1**, p. 233], is largely ignored by historians, mainly because Gregory doesn't seem to have realized that he has stumbled onto the single most important idea in calculus.

But what's important for mathematics educators is to find interesting and exemplary material that is accessible to students without a substantial number of prerequisites. The *Geometriae Pars Universalis* was seen even in its own time as a summary of the known techniques for solving calculus problems. (Even Newton's personal copy is said to be "dog–eared" [**6**, p. 29].) Consequently, it fills this need quite well, by providing examples of both modern analytic techniques and classical geometric techniques for solving calculus problems. Until now, the biggest problem with using the *Geometriae Pars Universalis* as an original source has been the lack of an English translation. However, a partial translation is now available online at the following link:

`http://math.knox.edu/aleahy/gregory/gpu.html`

References

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